December 1999
hep-th/0001031

STAR-PRODUCT AND MASSLESS FREE FIELD DYNAMICS IN AdS_4

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Abstract

Generic solution of free equations for massless fields of an arbitrary spin in AdS_4 is built in terms of the star-product algebra with spinor generating elements. A class of "plane wave" solutions is described explicitly.

1 Introduction

It has been shown (see [1] for a recent review) that the dynamics of massless fields of all spins in AdS_4 can be described in terms of star-product algebras acting on the auxiliary spinor variables. In this formalism the part of the (nonlinear) equations of motion that contain space-time derivatives has a form of zero-curvature or covariant constancy conditions and therefore can be solved explicitly at least locally. The aim of this paper is to illustrate how this machinery can be used in practice to derive solutions of the massless free field equations in AdS_4.

1.1 AdS Geometry

As is well-known, AdS_d geometry is described by the zero-curvature equations for the AdS_d algebra $o(d-1,2)$ with the gauge fields $A_{\underline{n}}{}^{AB}(x)$ ($A,B = 0 \div d$; $\underline{m},\underline{n} = 0 \div d-1$; $x^{\underline{n}}$ are coordinates of AdS_d) identified with the AdS_d gravitational fields according to

$$\omega_{\underline{n}}{}^{ab} = A_{\underline{n}}{}^{ab}\,, \qquad h_{\underline{n}}{}^{a} = \lambda^{-1} A_{\underline{n}}{}^{ad}\,, \tag{1.1}$$

where $a,b = 0 \div d-1$ and λ is a constant to be identified with the inverse AdS radius. The $o(d-1,2)$ field strengths are

$$R_{\underline{mn}}{}^{ab} = \partial_{\underline{m}}\omega_{\underline{n}}{}^{ab} + \omega_{\underline{m}}{}^{a}{}_{c}\,\omega_{\underline{n}}{}^{cb} - \lambda^2 h_{\underline{m}}{}^{a}\,h_{\underline{n}}{}^{b} - (\underline{m} \leftrightarrow \underline{n})\,, \tag{1.2}$$

$$R_{\underline{mn}}{}^{a} = \partial_{\underline{m}} h_{\underline{n}}{}^{a} + \omega_{\underline{m}}{}^{a}{}_{c} h_{\underline{n}}{}^{c} - (\underline{m} \leftrightarrow \underline{n})\,. \tag{1.3}$$

Interpreting $\omega_{\underline{n}}{}^{ab}$ as Lorentz connection and $h_{\underline{n}}{}^{b}$ as the frame 1-form one observes that $R_{\underline{mn}}{}^{a}$ identifies with the torsion tensor while the $\lambda-$independent part of $R_{\underline{mn}}{}^{ab}$ is the Riemann tensor. Setting $R_{\underline{mn}}{}^{a} = 0$ one expresses $\omega_{\underline{n}}{}^{ab}$ in terms of $h_{\underline{m}}{}^{a}$. Imposing the equation $R_{\underline{mn}}{}^{ab} = 0$ is then equivalent to the equation for AdS_d described in terms of the frame field $h_{\underline{n}}{}^{a}$ which is required to be non-degenerate. Thus, AdS_d geometry is described by the zero-curvature equation

$$R^{AB} = 0 \tag{1.4}$$

provided that $det|h_{\underline{n}}{}^{a}| \neq 0$.

From now on we focus on the particular case of AdS_4 using the well-known isomorphism $o(3,2) \sim sp(4|R)$. The algebra $sp(4|R)$ admits the oscillator realization with the generators

$$L_{\alpha\beta} = \frac{1}{4i}\{\hat{y}_\alpha, \hat{y}_\beta\}\,, \qquad \bar{L}_{\dot{\alpha}\dot{\beta}} = \frac{1}{4i}\{\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}\}\,, \qquad P_{\alpha\dot{\beta}} = \lambda\frac{1}{2i}\hat{y}_\alpha \hat{\bar{y}}_{\dot{\beta}} \tag{1.5}$$

realized as bilinears in two-component spinor oscillators satisfying the commutation relations[1]

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}\,, \qquad [\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad [\hat{y}_\alpha, \hat{\bar{y}}_{\dot{\beta}}] = 0 \tag{1.6}$$

($\alpha,\beta = 1,2,\ \dot{\alpha},\dot{\beta} = 1,2;\ \bar{y}_{\dot{\alpha}} = (y_\alpha)^+$; for conventions see Appendix).

The AdS_4 gravitational fields can now be identified with the 1-form bilinear in the oscillators

$$w_0 = dx^{\underline{n}} w_{0\underline{n}} = \frac{1}{4i} dx^{\underline{n}} (\omega_{0\underline{n}}{}^{\alpha\beta}\hat{y}_\alpha\hat{y}_\beta + \bar{\omega}_{0\underline{n}}{}^{\dot{\alpha}\dot{\beta}}\hat{\bar{y}}_{\dot{\alpha}}\hat{\bar{y}}_{\dot{\beta}} + 2\lambda h_{0\underline{n}}{}^{\alpha\dot{\beta}}\hat{y}_\alpha\hat{\bar{y}}_{\dot{\beta}}) \tag{1.7}$$

and satisfying the zero-curvature equation

$$0 = R_0 \equiv dw_0 - w_0 \wedge w_0\,. \tag{1.8}$$

Here $\omega_{0\underline{n}}{}^{\alpha\beta}(x)$ and $\bar{\omega}_{0\underline{n}}{}^{\dot{\alpha}\dot{\beta}}(x)$ describe Lorentz connection while $h_{0\underline{n}}{}^{\alpha\dot{\beta}}(x)$ describes vierbein in terms of two-component spinors.

[1]One can equivalently use the Majorana spinor oscillators $\hat{Y}_\nu$ ($\nu = 1 \div 4$) with the commutation relations $[\hat{Y}_\mu, \hat{Y}_\nu] = 2iC_{\mu\nu}$, where $C_{\mu\nu}$ is the charge conjugation matrix. The language of two-component spinors is however most useful for the analysis below.

The curvature $R = dw - w \wedge w$ admits the expansion analogous to (1.7) with the components

$$R_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_\alpha{}^\gamma \wedge \omega_{\beta\gamma} + \lambda^2 h_\alpha{}^{\dot\gamma} \wedge h_{\beta\dot\gamma}\,, \tag{1.9}$$

$$\bar R_{\dot\alpha\dot\beta} = d\bar\omega_{\dot\alpha\dot\beta} + \bar\omega_{\dot\alpha}{}^{\dot\gamma} \wedge \bar\omega_{\dot\beta\dot\gamma} + \lambda^2 h^\gamma{}_{\dot\alpha} \wedge h_{\gamma\dot\beta}\,, \tag{1.10}$$

$$R_{\alpha\dot\beta} = dh_{\alpha\dot\beta} + \omega_\alpha{}^\gamma \wedge h_{\gamma\dot\beta} + \bar\omega_{\dot\beta}{}^{\dot\gamma} \wedge h_{\alpha\dot\gamma}\,. \tag{1.11}$$

A particular solution of (1.8) can be chosen in the form

$$h_{0\underline{n}}{}^{\alpha\dot\beta} = -z^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta}\,, \tag{1.12}$$

$$\omega_{0\underline{n}}{}^{\alpha\alpha} = -\lambda^2 z^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta} x^\alpha{}_{\dot\beta}\,, \tag{1.13}$$

$$\bar\omega_{0\underline{n}}{}^{\dot\beta\dot\beta} = -\lambda^2 z^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta} x_\alpha{}^{\dot\beta}\,, \tag{1.14}$$

where

$$x_{\alpha\dot\beta} = \sigma^a_{\alpha\dot\beta} x_a\,, \qquad x^2 = x_a x^a = \frac{1}{2} x_{\alpha\dot\beta} x^{\alpha\dot\beta}\,, \qquad z = 1 + \lambda^2 x^2\,, \tag{1.15}$$

and sigma-matrices are Hermitian $\bar\sigma_{\underline{n}}{}^{\alpha\dot\beta} = \sigma_{\underline{n}}{}^{\beta\dot\alpha}$ with the normalization $\sigma_n{}^{\alpha\dot\beta}\sigma_{m\,\alpha\dot\beta} = 2\eta_{nm}$ where $\eta_{nm} = \{1, -1, -1, -1\}$ is the flat Minkowski metric. Following to [2] we use conventions with upper(lower) indices denoted by the same letter subject to symmetrization (see also Appendix).

The equations (1.12)-(1.14) describe the vierbein and Lorentz connection of AdS_4 corresponding to the stereographic coordinates for the hyperboloid realization of AdS_4 via embedding into the 5d flat space with the signature $(+,-,-,-,+)$. The AdS_4 metric tensor resulting from (1.12) is

$$g_{0\underline{mn}} = \frac{1}{2} h_{0\underline{m}}{}^{\alpha\dot\beta} h_{0\underline{n}\,\alpha\dot\beta} = \frac{\eta_{\underline{mn}}}{(1+\lambda^2 x^2)^2}\,. \tag{1.16}$$

Note that this form of the metric can be fixed by the requirement that it is conformally flat, regular at $x = 0$ and depends on the coordinates $x^{\alpha\dot\beta}$ via the Lorentz covariant combination x^2. In the flat limit $\lambda \to 0$ this metric describes the Minkowski space.

In these coordinates, the boundary of AdS_4 is identified with the hypersurface $z = 0$. The "north pole" is realized as infinity $x = \infty$. It becomes a regular point in the inversed coordinates

$$x^{\underline{n}} \to x_I^{\underline{n}} = \lambda^{-2}\frac{x^{\underline{n}}}{x^2} \tag{1.17}$$

with the gravitational fields of the form:

$$h^I_{0\underline{n}}{}^{\alpha\dot\beta} = -z_I^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta}\,, \tag{1.18}$$

$$\omega^I_{0\underline{n}}{}^{\alpha\alpha} = -\lambda^2 z_I^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta} x_I{}^\alpha{}_{\dot\beta}\,, \tag{1.19}$$

$$\bar\omega^I_{0\underline{n}}{}^{\dot\beta\dot\beta} = -\lambda^2 z_I^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta} x_{I\alpha}{}^{\dot\beta}\,, \tag{1.20}$$

This coordinate system describes the antipodal chart.

1.2 Star-Product and Free Massless Equations

Instead of working in terms of the operators $\hat{y}_\alpha$ and $\hat{\bar{y}}_{\dot\alpha}$ it is convenient to use the star-product defined by the formula

$$(f*g)(y,\bar{y}|x)=(2\pi)^{-4}\int d^2ud^2\bar{u}d^2vd^2\bar{v}f(y+u,\bar{y}+\bar{u}|x)g(y+v,\bar{y}+\bar{v}|x)e^{i(u_\alpha v^\alpha+\bar{u}_{\dot\alpha}\bar{v}^{\dot\alpha})} \tag{1.21}$$

It is elementary to see that this star-product is associative and well-defined for polynomial functions. The normalization is chosen such that 1 is the unit element of the algebra, i.e. $1*f=f*1=f$. The commutation relations for the generating elements y_α and $\bar{y}_{\dot\alpha}$ are

$$[y_\alpha,y_\beta]_*=2i\epsilon_{\alpha\beta}\,,\qquad [\bar{y}_{\dot\alpha},\bar{y}_{\dot\beta}]_*=2i\epsilon_{\dot\alpha\dot\beta}\,,\qquad [y_\alpha,\bar{y}_{\dot\beta}]_*=0\,, \tag{1.22}$$

where $[a,b]_*=a*b-b*a$. The formula (1.21) is equivalent to the standard differential Moyal star-product [3] by virtue of the Taylor expansion

$$f(y,\bar{y})=exp(y^\alpha\frac{\partial}{\partial z^\alpha}+\bar{y}^{\dot\alpha}\frac{\partial}{\partial\bar{z}^{\dot\alpha}})f(z,\bar{z})|_{z=\bar{z}=0}\,. \tag{1.23}$$

It describes the totally symmetric (i.e., Weyl) ordering of the operators $\hat{y}_\alpha,\hat{\bar{y}}_{\dot\alpha}$ in terms of symbols of operators and was called "triangle formula" in [4]. The star-product (1.21) acts on the auxiliary spinor variables $y_\alpha,\bar{y}_{\dot\alpha}$ rather than directly on the space-time coordinates as in the non-commutative Yang-Mills limit of the string theory [5]. It was argued however in [1] that the dynamical field equations for higher spin massless fields transform the star-product nonlocality in $y_\alpha,\bar{y}_{\dot\alpha}$ into the space-time nonlocality of the higher spin interactions.

In terms of the star-product, the equation of the background AdS_4 space has the form:

$$dw_0=w_0*\wedge w_0\,. \tag{1.24}$$

A less trivial fact shown in [2, 6] is that free equations for massless fields of all spins in AdS_4 can be cast into the form

$$dw_1-w_0*\wedge w_1-w_1*\wedge w_0=\frac{i}{4}[h_\alpha{}^{\dot\beta}\wedge h^{\alpha\dot\gamma}\frac{\partial}{\partial\bar{y}^{\dot\beta}}\frac{\partial}{\partial\bar{y}^{\dot\gamma}}C(0,\bar{y})+h^\alpha{}_{\dot\beta}\wedge h^{\gamma\dot\beta}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\gamma}C(y,0)], \tag{1.25}$$

$$dC=w_0*C-C*\tilde{w}_0\,. \tag{1.26}$$

Here $w_1(y,\bar{y}|x)$ and $C(y,\bar{y}|x)$ are functions of spinor and space-time coordinates and tilde is defined according to

$$\tilde{f}(y,\bar{y})=f(y,-\bar{y})\,. \tag{1.27}$$

Relativistic fields are identified with the coefficients in the Taylor expansions in powers of the auxiliary spinor variables

$$w_1(y,\bar{y}\mid x)=\sum_{n,m=0}^\infty\frac{\lambda^{1-|\frac{n-m}{2}|}}{2i\,n!m!}w_1^{\alpha_1\ldots\alpha_n}{}_,{}^{\dot\beta_1\ldots\dot\beta_m}(x)y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot\beta_1}\ldots\bar{y}_{\dot\beta_m} \tag{1.28}$$

and

$$C(y,\bar{y}|x)=\sum_{n,m=0}^\infty\frac{\lambda^{2-\frac{n+m}{2}}}{m!n!}C^{\alpha_1\ldots\alpha_n}{}_,{}^{\dot\alpha_1\ldots\dot\alpha_m}(x)y_{\alpha_1}\ldots y_{\alpha_n}\,\bar{y}_{\dot\alpha_1}\ldots\bar{y}_{\dot\alpha_m}\,, \tag{1.29}$$

Inserting (1.29) into (1.26), one arrives at the following infinite chain of equations

$$D^L C_{\alpha(m),\,\dot\beta(n)} = i h^{\gamma\dot\delta} C_{\alpha(m)\gamma,\,\dot\beta(n)\dot\delta} - i\lambda^2 nm h_{\alpha\dot\beta} C_{\alpha(m-1),\,\dot\beta(n-1)}\,, \tag{1.30}$$

where D^L is the Lorentz-covariant differential

$$D^L A_{\alpha\dot\beta} = dA_{\alpha\dot\beta} + \omega_\alpha{}^\gamma \wedge A_{\gamma\dot\beta} + \bar\omega_{\dot\beta}{}^{\dot\delta} \wedge A_{\alpha\dot\delta}\,. \tag{1.31}$$

The system (1.30) decomposes into a set of independent subsystems with $n-m$ fixed. It turns out [6] that the subsystem with $|n-m| = 2s$ describes a massless field of spin s (note that the fields $C_{\alpha(m),\,\dot\beta(n)}$ and $C_{\beta(n),\,\dot\alpha(m)}$ are complex conjugated).

Analogously, by the substitution of (1.28), the equation (1.25) amounts to

$$\begin{aligned} R_{1\,\alpha(n)\,,\dot\beta(m)} \equiv{}& D^L w_{1\,\alpha(n)\dot\beta(m)} + n\gamma(n,m,\lambda) h_\alpha{}^{\dot\delta} \wedge w_{1\,\alpha(n-1)\,,\dot\beta(m)\dot\delta} + m\gamma(m,n,\lambda) h^\gamma{}_{\dot\beta} \wedge w_{1\,\gamma\alpha(n)\,,\dot\beta(m-1)} \\ &= \delta(m) h^{\gamma\dot\delta} \wedge h^\gamma{}_{\dot\delta} C_{\alpha(n)\gamma(2)} + \delta(n) h^{\eta\dot\delta} \wedge h_\eta{}^{\dot\delta} \bar C_{\dot\beta(m)\dot\delta(2)}\,, \end{aligned} \tag{1.32}$$

where

$$\gamma(n,m,\lambda) = \theta(m-n) + \lambda^2\theta(n-m-2) + \lambda\delta(n-m-1). \tag{1.33}$$

A spin $s \geq 1$ dynamical massless field is identified with the 1-form (potential)

$$w_{\alpha(n),\dot\beta(n)} \qquad n = (s-1)\,, \qquad s \geq 1 \quad \text{integer}, \tag{1.34}$$

$$w_{\alpha(n),\dot\beta(m)} \qquad n+m = 2(s-1)\,, \quad |n-m| = 1 \qquad s \geq 3/2 \quad \text{half-integer}. \tag{1.35}$$

The matter fields are described by the 0-forms

$$C_{\alpha(0)\,,\dot\alpha(0)} \qquad s = 0\,, \tag{1.36}$$

$$C_{\alpha(1)\,,\dot\alpha(0)} \oplus C_{\alpha(0)\,,\dot\alpha(1)} \qquad s = 1/2\,. \tag{1.37}$$

All other components of the expansions (1.28) and (1.29) express in terms of the derivatives of physical fields by virtue of the equations (1.32), (1.30). For example, for 0-forms C one has

$$C_{\alpha(n)\,,\dot\beta(m)} = \frac{1}{(2i)^{\frac12(n+m-2s)}} h^{\underline{n}_1}_{\alpha\dot\beta} D^L_{\underline{n}_1} \ldots h^{\underline{n}_{\frac12(n+m-2s)}}_{\alpha\dot\beta} D^L_{\underline{n}_{\frac12(n+m-2s)}} C_{\alpha(2s)} \qquad n \geq m, \tag{1.38}$$

or

$$C_{\alpha(n)\,,\dot\beta(m)} = \frac{1}{(2i)^{\frac12(n+m-2s)}} h^{\underline{n}_1}_{\alpha\dot\beta} D^L_{\underline{n}_1} \ldots h^{\underline{n}_{\frac12(n+m-2s)}}_{\alpha\dot\beta} D^L_{\underline{n}_{\frac12(n+m-2s)}} C_{\dot\beta(2s)} \qquad n \leq m. \tag{1.39}$$

Analogous formula holds for the 1-forms w

$$w_{\alpha(n)\,,\dot\beta(m)} \sim \left(\frac{\partial}{\partial x}\right)^{[\frac{|n-m|}{2}]} w^{phys} + w^{gauge}\,, \tag{1.40}$$

where w^{phys} denotes the appropriate field from the list (1.34), (1.35) while w^{gauge} is a pure gauge part.

As shown in [2, 6] the content of the equations (1.25) and (1.26) is equivalent to the relations (1.32), (1.30) and usual dynamical equations for the physical massless fields in AdS_4. The fields

$$w^{\alpha(n),\dot{\beta}(m)} \quad \text{with} \quad n+m=2(s-1) \tag{1.41}$$

and

$$C_{\alpha(m),\dot{\beta}(n)} \quad \text{with} \quad |n-m|=2s \tag{1.42}$$

are associated with the massless field of spin s along with all its on-mass-shell nontrivial derivatives. For spin $s \geq 1$, the 0 forms C describe gauge invariant field strengths generalizing the spin 1 Maxwell field strength and spin 2 Weyl tensor to an arbitrary spin. For example, in the spin 2 case (1.28) is equivalent to the linearized Einstein equations because it just tells us that torsion is zero and all the components of the linearized Riemann tensor are zero except for those which are described by the Weyl tensor $C_{\alpha_1\ldots\alpha_4}$, $C_{\dot{\alpha}_1\ldots\dot{\alpha}_4}$.

The aim of this paper is to explore the fact that once the dynamical equations are reformulated in the form (1.25) and (1.26) one can write down their generic solution explicitly in terms of the star-product provided that the AdS_4 vacuum equations (1.24)-(1.26) are solved in the pure gauge form

$$w_0 = -g^{-1} * dg\,, \tag{1.43}$$

where $g(y,\bar{y}|x)$ is some invertible element of the star-product algebra, $g*g^{-1} = g^{-1}*g = 1$ and $d = dx^{\underline{n}}\frac{\partial}{\partial x^{\underline{n}}}$ is the space-time differential. A form of the appropriate $g(y,\bar{y}|x)$ is found in the section 2. The covariant constancy equation (1.26) has the generic solution of the form

$$C(x) = g^{-1}(x) * C_0 * \tilde{g}(x)\,, \tag{1.44}$$

where $C_0(y,\bar{y})$ is an arbitrary $x-$independent element of the star-product algebra. For exponential functions C_0 we reproduce in the section 3 a set of particular solutions that in the flat limit tend to the plane wave solutions for massless fields of an arbitrary spin. A less trivial fact shown in the section 4 is that it is also possible to write down a generic solution of the equation (1.25) despite it does not have a zero-curvature form.

Let us emphasize that this way of solving dynamical equations reduces to evaluation of some elementary Gaussian integrals originating from the star-product (1.21) (i.e. there is no need to solve any differential equations). The equation (1.44) can be interpreted as a covariantized Taylor expansion. Indeed, if $g(x_0) = 1$ for some x_0 then $C_0(y,\bar{y}) = C(y,\bar{y}|x_0)$. In accordance with (1.29), (1.38) and (1.39), $C_0(y,\bar{y})$ therefore describes all on-mass-shell nontrivial derivatives of the physical field at the point x_0.

2 Gauge Function

In this section we find the function $g(y,\bar{y}|x)$ that reproduces the AdS_4 gravitational fields (1.12)-(1.14) in the star-product algebra version of the ansatz (1.7)

$$w_{0\underline{n}} = \frac{1}{4i}(\omega_{0\underline{n}}{}^{\alpha\beta}y_\alpha y_\beta + \bar{\omega}_{0\underline{n}}{}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} + 2\lambda h_{0\underline{n}}{}^{\alpha\dot{\beta}}y_\alpha\bar{y}_{\dot{\beta}}) \tag{2.1}$$

by virtue of the pure gauge representation (1.43). Note that the bilinear ansatz is consistent, because the bilinears form a closed $sp(4)$ subalgebra with respect to commutators[2]. We therefore have to find such a function g that the resulting connection (1.43) is bilinear in the auxiliary oscillators y and $\bar{y}$. Note that if w_0 would contain some higher-order polynomials in oscillators, this would imply that some higher spin fields acquire nonvanishing vacuum expectation values thus making the physical interpretation of the corresponding solutions less straightforward.

Let us look for the solution for g in the Lorentz-covariant form

$$g(y,\bar{y}|x) = e^{if(x^2)x^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}+r(x^2)} \tag{2.2}$$

with some functions $f(x^2)$ and $r(x^2)$. One readily shows that the inverse element g^{-1} is

$$g^{-1}(y,\bar{y}|x) = (1+f^2x^2)^2 e^{-if(x^2)x^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}-r(x^2)}\,. \tag{2.3}$$

Direct computation using the star-product (1.21) leads by virtue of evaluation of elementary Gaussian integrals to the following result

$$g^{-1}*dg = 2(m_1+m_2x^{\alpha\dot{\beta}}y_\alpha\bar{y}_{\dot{\beta}})x_n dx^n + m_3 y_\alpha\bar{y}_{\dot{\beta}}dx^{\alpha\dot{\beta}} + m_4(x^\gamma{}_{\dot{\beta}}y_\alpha y_\gamma + x_\alpha{}^{\dot{\gamma}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\gamma}})dx^{\alpha\dot{\beta}}\,, \tag{2.4}$$

where

$$\begin{aligned} m_1 &= r' - q^{-1}(2ff'x^2+f^2),\\ m_2 &= iq^{-1}f' + iq^{-2}f^3\,,\\ m_3 &= iq^{-2}f(1-f^2x^2)\,,\\ m_4 &= iq^{-2}f^2\,, \end{aligned}$$

and we use notation

$$q = 1+f^2x^2$$

and

$$h'(x^2) = \frac{\partial h(x^2)}{\partial x^2}\,.$$

In order to reproduce AdS_4 vierbein and connection (1.12)-(1.14) we have to set $m_1 = m_2 = 0$. The condition $m_2 = 0$ is equivalent to

$$f^3 + f' + f'f^2x^2 = 0. \tag{2.5}$$

Its general solution is

$$f_\pm = \lambda(\frac{1}{1\pm\sqrt{z}})\,, \tag{2.6}$$

where λ is an arbitrary integration constant and z is defined in (1.15). The solution f_+ is regular at $x\to 0$. The solution f_- is singular at $x\to 0$. In fact, the two gauge functions g corresponding to these solutions are related to each other by the inversion (1.17) away

[2] It is a simple exercise with the star-product to check that this ansatz with the fields (1.12)-(1.14) satisfies (1.24)

from the north ($x=\infty$) and south ($x=0$) poles thus solving the problem for the two stereographic charts. In the rest of the paper we will for definiteness consider the solution regular at $x=0$ with $f=f_+$.

Solving the condition $m_1=0$ for f_+, one finds up to the integration constant that does not affect w_0, that

$$r = ln(\frac{2\sqrt{z}}{1+\sqrt{z}}). \tag{2.7}$$

As a result g takes the form

$$g(y,\bar{y}|x) = 2\frac{\sqrt{z}}{1+\sqrt{z}}\exp[\frac{i\lambda}{1+\sqrt{z}}x^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}] \tag{2.8}$$

with the inverse

$$g^{-1}(y,\bar{y}|x) = \tilde{g}(y,\bar{y}|x) = 2\frac{\sqrt{z}}{1+\sqrt{z}}\exp[\frac{-i\lambda}{1+\sqrt{z}}x^{\alpha\dot{\beta}}y_\alpha\bar{y}_{\dot{\beta}}]\,. \tag{2.9}$$

Inserting (2.6) and (2.7) into the expressions for m_3 and m_4 we find that the corresponding Lorentz connection and vierbein reproduce (1.12)-(1.14). The solution (1.18)-(1.20) corresponds to f_-.

3 Plane Waves

Having found the gauge function g one solves the free field equation (1.26) for matter fields and Weyl tensors of an arbitrary spin in the form (1.44) equivalent to

$$C(y,\bar{y}|x) = (2\pi)^{-4}\int_{-\infty}^{\infty} d^2u d^2v d^2\bar{u} d^2\bar{v}\exp i(u_\alpha v^\alpha + \bar{u}_{\dot{\alpha}}\bar{v}^{\dot{\alpha}})$$
$$\times g^{-1}(y+u,\bar{y}+\bar{u})C_0(y+u+v,\bar{y}+\bar{u}+\bar{v})\tilde{g}(y+v,\bar{y}+\bar{v})\,. \tag{3.1}$$

As emphasized in the section 1.2, this formula reproduces the covariantized Taylor expansion with C_0 identified with all on-mass-shell nontrivial derivatives of $C(x)$ at $x=x_0$ with $g(x_0)=I$. Note that $x_0=0$ is the "south pole" for the solution (2.8).

Let us now choose C_0 in the form

$$C_0 = \lambda^2 c_0 \exp i\lambda^{-\frac{1}{2}}(y^\alpha\eta_\alpha + \bar{y}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}})\,, \tag{3.2}$$

where c_0 is an arbitrary constant and η_α and $\bar{\eta}_{\dot{\alpha}}$ are complex conjugated constant spinor parameters. Substitution to (3.1) leads to the following result upon evaluation of the elementary Gaussian integrals

$$C(y,\bar{y}|x) = c_0\lambda^2 z\exp i\left[-(\lambda y_\alpha\bar{y}_{\dot{\beta}} + \eta_\alpha\bar{\eta}_{\dot{\beta}})x^{\alpha\dot{\beta}} + \lambda^{-\frac{1}{2}}\sqrt{z}(y^\alpha\eta_\alpha + \bar{y}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}})\right]\,. \tag{3.3}$$

Taking into account the identification of the particular components of the expansion (1.29) explained in the section 1.2 one identifies the matter fields and higher spin Weyl tensors with

$$C_{\alpha_1\ldots\alpha_n}(x) = \lambda^{\frac{n}{2}-2}\frac{\partial}{\partial y^{\alpha_1}}\ldots\frac{\partial}{\partial y^{\alpha_n}}C(y,\bar{y}|x)|_{y=\bar{y}=0} \tag{3.4}$$

and

$$\bar{C}_{\dot{\alpha}_1\ldots\dot{\alpha}_n}(x) = \lambda^{\frac{n}{2}-2}\frac{\partial}{\partial \bar{y}^{\dot{\alpha}_1}}\ldots\frac{\partial}{\partial \bar{y}^{\dot{\alpha}_n}}C(y,\bar{y}|x)|_{y=\bar{y}=0}\,. \tag{3.5}$$

From (3.3) one gets

$$C_{\alpha\ldots\alpha_{2s}}(x) = c_0 z^{s+1}\eta_{\alpha_1}\ldots\eta_{\alpha_{2s}}\exp ik_{\gamma\dot{\beta}}x^{\gamma\dot{\beta}} \tag{3.6}$$

and

$$\bar{C}_{\dot{\alpha}\ldots\dot{\alpha}_{2s}}(x) = c_0 z^{s+1}\bar{\eta}_{\dot{\alpha}_1}\ldots\bar{\eta}_{\dot{\alpha}_{2s}}\exp ik_{\gamma\dot{\beta}}x^{\gamma\dot{\beta}}\,, \tag{3.7}$$

where

$$k_{\alpha\dot{\beta}} = -\eta_\alpha\bar{\eta}_{\dot{\beta}}\,. \tag{3.8}$$

The expression (3.8) is the standard twistor representation [7] for an arbitrary light-like vector $k_{\alpha\dot{\beta}}$ in terms of spinors.

By construction, the formulae (3.6) and (3.7) describe solutions of the free equations of motion of massless fields of arbitrary spin in AdS_4. In the flat space $\lambda \to 0$ (i.e., $z \to 1$) these formulae describe usual flat space plane waves. We therefore interpret the solution (3.3) as describing AdS_4 "plane waves". Let us note that, as expected, these solutions tend to zero at the boundary of AdS_4 $z = 0$.

4 Higher Spin Potentials

Once the 0-forms C are found one can in principle solve the equation (1.25) for the gauge potentials w_1 modulo gauge transformation

$$\delta w_1 = d\epsilon - w_0 * \epsilon + \epsilon * w_0. \tag{4.1}$$

where w_0 is the background AdS_4 gauge (gravitational) field (2.1) and $\epsilon(y,\bar{y}|x)$ is an arbitrary gauge parameter.

This problem is analogous to the reconstruction of the electromagnetic potential via the field strength or the metric tensor via the Riemann tensor. Since the equation (1.25) is formally consistent it admits some solution. Remarkably, this solution can also be found explicitly for the general gauge function $g(y,\bar{y}|x)$ and $C_0(y,\bar{y})$. The systematic derivation can be obtained with the help of a more sophisticated technics developed in [8, 1] for the analysis of the nonlinear problem. Here we announce the final result

$$\begin{aligned} w_1(y,\bar{y}|x) \;=\; & \frac{1}{32\pi^4}({\omega_0}^{\alpha\beta}\frac{\partial}{\partial y^\beta} + \lambda {h_0}^{\alpha\dot{\beta}}\frac{\partial}{\partial \bar{y}^{\dot{\beta}}})\int_0^1 sds\int_{-\infty}^{\infty} d^2u d^2v d^2\bar{u} d^2\bar{v}[(y_\alpha + 2v_\alpha) \\ & \times g^{-1}((1-s)y + u, \bar{y} + \bar{u}|x) C_0((1-s)y + v + u, \bar{y} + \bar{u} + \bar{v}) \\ & \times g((1-s)y + (1-2s)v, \bar{y} + \bar{v}|x)\exp i(uv + \bar{u}\bar{v})] + c.c.\,, \end{aligned} \tag{4.2}$$

leaving the details of its derivation to a more detailed future publication [9].

Note that this expression admits no simple interpretation in terms of the star-product (1.21). This happens because it is derived [9] as some projection from a larger star-product algebra. However, one can check directly that (4.2) solves (1.25). To this end one inserts this formula into the left hand side of (1.25) using the equations (1.26). Then one observes that the resulting terms in the integrand sum up to a total derivative in the integration variable s. The contribution at $s = 0$ vanishes because of the factor of s in the integration measure, while the contribution at $s = 1$ reproduces the right hand side of (1.25). Note that only the terms independent of the Lorentz connection contribute. The computation sketched above is however tedious enough.

Insertion of the expression (3.2) into (4.2) leads to the following result

$$\begin{aligned} w_1 = & \frac{1}{2} z c_0 (\omega^{\alpha\beta} \frac{\partial}{\partial y^\beta} + \lambda h^{\alpha\dot{\gamma}} \frac{\partial}{\partial \bar{y}^{\dot{\gamma}}}) \int_0^1 ds \frac{s}{(s + (1-s)\sqrt{z})^3} \\ & \times (y_\alpha + \lambda^{-1/2}(1+\sqrt{z})\eta_\alpha + \lambda^{1/2} x_{\alpha\dot{\beta}} \bar{\eta}^{\dot{\beta}} - \lambda \bar{y}^{\dot{\beta}} x_{\alpha\dot{\beta}}) \\ & \times \exp i \frac{\lambda^{-1/2}}{s + (1-s)\sqrt{z}} \Big[(1-s)(\eta_\alpha y^\alpha + \bar{\eta}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}} + \lambda x^{\alpha\dot{\beta}} (y_\alpha \bar{\eta}_{\dot{\beta}} + \eta_\alpha \bar{y}_{\dot{\beta}})) \\ & \qquad + s(\sqrt{z} \bar{\eta}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}} - \lambda^{1/2} x^{\alpha\dot{\beta}} \eta_\alpha \bar{\eta}_{\dot{\beta}}) \Big] + c.c. \end{aligned} \tag{4.3}$$

Let us note that the quantity $s + (1-s)\sqrt{z}$ is strictly positive in the region $z > 0$ for any $0 \le s \le 1$. Therefore, the right hand side of the formula (4.3) is some entire function in the auxiliary spinor variables y and $\bar{y}$. This means that it gives rise to the well-defined gauge potentials associated with the coefficients of the expansion (1.28) everywhere in the north pole chart. The south pole chart can be analyzed analogously. In the limit $z \to 0$ the expression (4.3) acquires singularity at $s = 0$.

5 Conclusion

It is demonstrated that the general formalism developed originally for the formulation of the interacting higher spin gauge theories allows one to write down explicit solutions of the field equations in terms of the star-product algebras in auxiliary spinor variables. The true reason for this is that the formalism of star-product algebras makes infinite-dimensional higher spin symmetries explicit thus allowing one to formulate the field equations as certain covariant constancy conditions. For simplicity, in this paper we have focused on the most symmetric case with gravitational fields possessing explicit Lorentz symmetry. The developed approach can be applied in other coordinate systems. We believe that it will have a wide area of applicability and can be extended to dynamical systems in various dimensions, black hole and brane backgrounds as well as to the superspace.

Acknowledgments

The authors are grateful to R.Metsaev, S.Konstein and D.Sorokin for useful conversations. This research was supported in part by INTAS, Grant No.96-0308 and by the RFBR Grant No.99-02-16207.

Appendix. Notation

Following to [2], we use conventions with upper(lower) indices denoted by the same letter subject to symmetrization prior possible contractions with lower(upper) indices denoted by the same letter. With these conventions only a number of indices within any symmetrized group is important. This number is often indicated in brackets. A maximal possible number of lower and upper indices denoted by the same letter is supposed to be contracted.

We choose mostly minus flat Minkowsi metric $\eta^{nm} = \{1, -1, -1, -1\}$. $\partial_{\underline{n}} = \frac{\partial}{\partial x^{\underline{n}}}$.

Two-component spinor indices are raised and lowered according to:

$$A_\alpha = \epsilon_{\beta\alpha} A^\beta \qquad A^\alpha = \epsilon^{\alpha\beta} A_\beta \,,$$

where $e_{12} = e^{12} = 1$. Sigma matrices are expressed in terms of Pauli matrices according to

$$\sigma^m_{\alpha\dot{\beta}} = (I, \sigma^i)$$

$$\sigma^n_{\alpha\dot{\beta}} \sigma^{m\alpha\dot{\beta}} = 2\eta^{nm} \,.$$

Also the following conventions are used

$$x_{\alpha\dot{\beta}} = \sigma^a_{\alpha\dot{\beta}} x_a \,, \qquad x^2 = x_a x^a = \frac{1}{2} x_{\alpha\dot{\beta}} x^{\alpha\dot{\beta}}.$$

$$\delta(n) = \begin{cases} 1\,, \; n = 0 \\ 0\,, \; n \neq 0 \end{cases} ,$$

$$\theta(n) = \begin{cases} 1\,, \; n \geq 0 \\ 0\,, \; n < 0 \end{cases} .$$

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